

On Siegel threefolds with a projective Calabi–Yau model

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Introduction

In the papers [FS1], [FS2] we described some Siegel modular threefolds which admit a weak Calabi–Yau model.*) Not all of them admit a *projective* model. In fact, Bert van Geemen, in a private communication, pointed out a significative example which cannot admit a projective model. His comment was a starting motivation for this paper. We mention that a weak Calabi–Yau threefold is projective if, and only if, it admits a Kaehler metric. The purpose of this paper is to exhibit criterions for the projectivity, to treat several examples, and to compute their Hodge numbers.

Basic for our examples is a certain complete intersection \mathcal{X} of four quadrics introduced the paper [GN] of van Geemen and Nygaard:

$$\begin{aligned} \mathcal{X} : \quad & Y_0^2 = X_0^2 + X_1^2 + X_2^2 + X_3^2, \\ & Y_1^2 = X_0^2 - X_1^2 + X_2^2 - X_3^2, \\ & Y_2^2 = X_0^2 + X_1^2 - X_2^2 - X_3^2, \\ & Y_3^2 = X_0^2 - X_1^2 - X_2^2 + X_3^2 \end{aligned}$$

The variety \mathcal{X} has 96 isolated singularities which are ordinary double points (nodes). One of them, called the *standard node*, is

$$\eta = [\sqrt{2}, 0, \sqrt{2}, 0, 1, 1, 0, 0].$$

In the paper [CM] it has been pointed out that the results of [GN] imply that \mathcal{X} admits a resolution that is a (projective) Calabi–Yau threefold. The

*) In this paper a weak Calabi–Yau threefold is understood as a compact complex threefold of whose first Betti number vanishes and which admits an everywhere holomorphic differential form of degree 3 without zeros.

holomorphic three form without zeros (unique up to a constant) on this model is given by

$$\frac{X_2^4}{Y_0 Y_1 Y_2 Y_3} d(X_0/X_2) \wedge d(X_1/X_2) \wedge d(X_3/X_2).$$

The basic result – essentially due to van Geemen and Nygaard [GN] – is the following theorem.

Theorem. *The Hodge numbers of a Calabi–Yau desingularization of \mathcal{X} are*

$$h_{11} = 32, \quad h_{12} = 0.$$

Hence this Calabi–Yau manifold is rigid.

We recall that the two Hodge numbers and the Euler number e are related by the formula

$$e = 2(h_{11} - h_{12}),$$

hence the Euler number for this example is $e = 64$.

In [FS2] we introduced a finite group $\mathcal{G} \subset \mathrm{GL}(8, \mathbb{C})$ which acts on \mathcal{X} . It is generated by the four transformations

$$\begin{aligned} U_1 : & (Y_0, Y_1, Y_3, Y_2, X_0, X_3, X_2, X_1), \\ U_2 : & (Y_0, Y_3, Y_2, Y_1, X_0, X_1, X_3, X_2), \\ T : & (Y_0, -iY_1, Y_2, -iY_3, X_1, X_0, X_3, X_2), \\ J : & \sqrt{2} \cdot (X_0, X_1, X_2, X_3, Y_0/2, Y_1/2, Y_2/2, Y_3/2). \end{aligned}$$

This group contains the subgroup \mathcal{Z} of order 4 generated by multiplication with i . The group

$$\bar{\mathcal{G}} := \mathcal{G}/\mathcal{Z}$$

acts faithfully on \mathcal{X} . Its order is $24\,576 = 2^{13} \cdot 3$. There is a subgroup $\mathcal{H} \subset \mathcal{G}$ of index two which leaves the holomorphic three form invariant, namely, the group generated by

$$U_1 U_2, \quad U_1 T, \quad U_2 T, \quad J.$$

It also contains \mathcal{Z} and

$$\bar{\mathcal{H}} := \mathcal{H}/\mathcal{Z}$$

is a group of order $12\,288 = 2^{12} \cdot 3$ which acts faithfully on \mathcal{X} and which leaves the holomorphic 3-form invariant. In [FS2] has been proved:

Theorem. *Let G be any subgroup of $\bar{\mathcal{H}}$. Then there exists a desingularization (in the category of complex spaces) of \mathcal{X}/G that is a weak Calabi–Yau threefold.*

Actually we proved more in [FS2], namely that the varieties \mathcal{X}/G are Siegel threefolds.

There are 4117 conjugacy classes of subgroups of $\bar{\mathcal{H}}$ so, in principle, we get 4117 examples of weak Calabi–Yau threefolds. Of course there might be biholomorphic equivalent ones under them, but, this may be difficult to decide.

Let $G \subset \bar{\mathcal{H}}$ be a subgroup. One can ask whether there exists a resolution of \mathcal{X}/G in the form of a *projective* Calabi–Yau manifold. In all cases that we are able to treat we will construct first a resolution $\tilde{\mathcal{X}} \rightarrow \mathcal{X}$ such that G extends to a group of biholomorphic self maps of $\tilde{\mathcal{X}}$. After that we will construct a resolution of $\tilde{\mathcal{X}}/G$ in the form of a weak Calabi–Yau manifold.

We explain the construction of $\tilde{\mathcal{X}}$. For this we have to consider two classes \mathcal{A}, \mathcal{B} of nodes of \mathcal{X} .

A node a belong to the class \mathcal{A} if its stabilizer G_a contains an element g that is conjugate to the transformation

$$(Y_0, \dots, X_3) \longmapsto (Y_0, -Y_1, Y_2, -Y_3, X_0, X_1, -X_2, -X_3).$$

This transformation occurred already in [FS2]. It has the following property. Consider the blow up of the node. The exceptional divisor ($\cong \mathbb{P}^1 \times \mathbb{P}^1$) is in the fixpoint locus of g .

Next we explain the second class \mathcal{B} . For this one needs the small resolutions of the node. A resolution is called *small* if the exceptional set is a curve. One knows that a node admits two (isomorphism classes) of small resolutions. By a *ruling* of the node we understand the choice of one of the two small resolutions. There is a subgroup of index two of $\bar{\mathcal{G}}_a$ that preserves the rulings. We described this subgroup in [FS2]. This subgroup extends to the small resolutions of the node. By definition, the class \mathcal{B} consists of all nodes a such that the elements of the stabilizer G_a preserve the rulings of the node. Assume that $\mathcal{A} \cup \mathcal{B}$ is the set of all nodes. Then we can construct a resolution $\tilde{\mathcal{X}} \rightarrow \mathcal{X}$ as follows. For the nodes in \mathcal{A} we take the blow up and for the rest we take a small resolution. This can be done in such a way that G acts on $\tilde{\mathcal{X}}$ as group of biholomorphic transformations. Now a general theorem, essentially due to Roan [Ro], shows the following result. (compare [FS2], Theorem 1.5).

Theorem. *Let $G \subset \bar{\mathcal{H}}$ be a subgroup such that each node is contained in $\mathcal{A} \cup \mathcal{B}$. Then there exists a resolution $\tilde{\mathcal{X}} \rightarrow \mathcal{X}$ in the category of complex spaces such that the action of G extends to $\tilde{\mathcal{X}}$ and such that $\tilde{\mathcal{X}}/G$ admits a resolution in the form of a weak Calabi–Yau threefold. There are 2791 conjugacy classes of subgroups of $\bar{\mathcal{H}}$ with this property.*

Now there arises the question whether $\tilde{\mathcal{X}}$ can be obtained as *projective variety*. If this is the case then we can construct the resolution $\tilde{\mathcal{X}}/G$ on the form of the Hilbert scheme $G\text{-Hilb}(\tilde{\mathcal{X}})$ as a (projective) Calabi–Yau manifold (see [BKR], compare also [FS1], Theorem 2.6). We will derive a projectivity criterion to obtain the following result.

Theorem. *Let $G \subset \bar{\mathcal{H}}$ be a group such that $\mathcal{A} \cup \mathcal{B}$ is the set of all nodes. Assume that for each node $a \in \mathcal{B}$, $a \notin \mathcal{A}$, there exists a divisor on \mathcal{X} that runs in a non trivial way into the node. Then $\tilde{\mathcal{X}}$ can be constructed as projective manifold. As a consequence, there exists a resolution of \mathcal{X}/G in the form of a (projective) Calabi-Yau manifold. There are 1986 conjugacy classes of subgroups of $\bar{\mathcal{H}}$ with this property.*

Here “non-trivial” means that the divisor is not the divisor of a meromorphic function in any small (analytic) open neighborhood of the node. To prove projectivity, one needs information about the map $\text{Cl}(\mathcal{X}) \rightarrow \text{Cl}(\mathcal{X}, a)$. Here $\text{Cl}(\mathcal{X})$ is the group of divisor classes of \mathcal{X} , and the group $\text{Cl}(\mathcal{X}, a)$ is the limit of the analytic divisor class groups of small neighborhoods of a . We already have seen in [FS2] that $\text{Cl}(\mathcal{X})$ is a group of rank 32. In Sect. 2 we describe an explicit system of generators. The group $\text{Cl}(\mathcal{X}, a)$ is isomorphic to \mathbb{Z} for a node. A projective small resolution of a node can be obtained if one blows up a divisor that runs non-trivially into the node which means that the image in $\text{Cl}(\mathcal{X}, a)$ is different from 0. Hence we have to describe the images of the generating divisor in $\text{Cl}(\mathcal{X}, a)$. This is done in Sects. 3–5 and needs analytic methods. Actually we use the description of van Geemen and Nygaard as modular variety. The nodes correspond to certain 0-dimensional cusps. We develop a theory of local Borcherds products to describe $\text{Cl}(\mathcal{X}, a)$. This theory enables us to compute the map $\text{Cl}(\mathcal{X}) \rightarrow \text{Cl}(\mathcal{X}, a)$ explicitly. It looks rather involved to introduce a new theory of local Borcherds products in this context. But this theory may be of interest in its own right.

In the last three sections we show how the Hodge numbers of many examples can be computed. Many examples rest on computer calculations. Nevertheless we treated some examples in detail where the calculations can be done by hand.

1. The divisor class group

In the paper [FS2] we investigated the divisor class group $\text{Cl}(\mathcal{X})$ and proved that $\text{Cl}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ has dimension 32. Generators have been found using the decomposition of the divisor of Igusa’s modular form χ_{35} into irreducible components. We reproduce these results in a modified form. For this we start with the three forms

$$X_2 - X_3, \quad X_0X_2 + X_1X_3, \quad X_0 - X_1 - X_2 - X_3.$$

Their zero divisors on \mathcal{X} are not irreducible. Each of them can be decomposed into two divisors as follows:

$$D_1^{\pm} : \quad X_2 - X_3 = Y_1 \pm Y_3 = 0,$$

$$D_2^{\pm} : \quad X_0X_2 + X_1X_3 = Y_0Y_1 \pm Y_2Y_3 = 0,$$

$$D_3^{\pm} : \quad X_0 - X_1 - X_2 - X_3 = Y_1(X_1 + X_3) \pm (\sqrt{2}/2)Y_2Y_3.$$

To be precise we mention that this is only a set theoretical description. The precise definition of $-$ for example D_3^+ is that the associated ideal the divisor is the radical of the image of the ideal

$$(X_0 - X_1 - X_2 - X_3, Y_1(X_1 + X_3) + (\sqrt{2}/2)Y_2Y_3)$$

in the coordinate ring of \mathcal{X} (factor ring of $\mathbb{C}[Y_0, \dots, X_3]$ by the defining ideal). We will make use of the orbits of the divisors D_i^\pm under \mathcal{G} .

1.1 Lemma. *For every $i \in \{1, 2, 3\}$ the divisor D_i^+ and D_i^- are in the same \mathcal{G} -orbit. The order of the orbit of the divisors D_i^\pm under \mathcal{G} is 48 for D_1^\pm , 12 for D_2^\pm , and 128 for D_3^\pm .*

From [FS2] we recall the result that $\text{Cl}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the direct sum of \mathcal{G} -irreducible subspaces of dimension 1, 3, 12, 16. We consider the subspaces of $\text{Cl}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by the orbits of D_i . Each of this three subspace contains the trivial one-dimensional representation. The results described in [FS2] imply the following proposition.

1.2 Proposition. *Consider the factor space of $\text{Cl}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ by the one-dimensional representation. Its three irreducible components can be described as the \mathcal{G} -submodules generated by the divisors D_i^\pm . More precisely, D_1^\pm generates the subspace of dimension 12, D_2^\pm generates the subspace of dimension 3, and D_3^\pm generates the subspace of dimension 16.*

We want to describe how the generating divisors run into the standard node. For this we need information about its stabilizer. It has been described in [FS2].

1.3 Lemma. *The following transformations stabilize the standard node and preserves its ruling:*

$$\begin{aligned} &(Y_0, Y_3, Y_2, Y_1, X_0, X_1, X_3, X_2), \\ &(Y_0, iY_1, Y_2, iY_3, X_1, X_0, X_3, X_2), \\ &(Y_2, Y_3, Y_0, Y_1, X_0, X_1, -iX_2, -iX_3), \\ &(Y_0, Y_1, Y_2, -Y_3, X_0, X_1, X_2, -X_3), \\ &(Y_0, -Y_1, Y_2, -Y_3, X_0, X_1, X_2, X_3), \\ &(Y_0, Y_1, Y_2, Y_3, X_0, X_1, -X_2, -X_3), \\ &(2X_0, 2X_2, 2X_1, 2X_3, Y_0, Y_2, Y_1, Y_3). \end{aligned}$$

Their images in $\bar{\mathcal{G}}$ generate a subgroup order $128 = 2^7$ which is a subgroup of index two of the stabilizer.

Now we consider the set of all basic divisors that run into the standard node and decompose it into orbits.

1.4 Theorem. *Consider the set of 188 divisors described in Lemma 1.1. Let \mathcal{S} be the subset of divisors that run into the standard node. The group of all*

elements of \mathcal{G} that stabilize the standard node and its rulings (see 1.3) acts on this set. There are six orbits that can be represented by the six divisors

$$D_1^+, D_1^-, D_2^+, D_2^-, D_3^+, D_3^-.$$

In the next sections we will use this result to determine the classes of the basic divisors in small neighborhood of a node.

2. The local divisor class group of a node

Let M be a locally compact space. An open subset U is called a neighborhood of ∞ if its complement is a compact subset. We define

$$H_\infty^q(M, \mathbb{Z}) := \varprojlim H^q(U, \mathbb{Z})$$

as the limit of the cohomology groups of all open neighborhoods of ∞ . There is an exact sequence

$$\cdots \longrightarrow H_c^q(M, \mathbb{Z}) \longrightarrow H^q(M, \mathbb{Z}) \longrightarrow H_\infty^q(M, \mathbb{Z}) \longrightarrow H_c^{q+1}(M, \mathbb{Z}) \longrightarrow \cdots$$

We mention that it is sufficient that U runs through a fundamental system of open neighborhoods of ∞ .

We apply this to the tangent bundle M of the sphere S_n for $n > 1$. Explicitly, this is

$$M = \left\{ (x, y) \in \mathbb{R}^{2(n+1)}; \quad \sum x_i^2 = 1, \sum x_i y_i = 0 \right\}.$$

Since this is homotopically equivalent to the sphere, we have

$$H^q(M, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } q = 0 \text{ or } q = n, \\ 0 & \text{else.} \end{cases}$$

Poincarè duality gives (for the manifold M of dimension $2n$)

$$H_c^q(M, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } q = n \text{ or } q = 2n, \\ 0 & \text{else.} \end{cases}$$

If $H^q(M, \mathbb{Z})$ is different from 0, its generator has not compact support. Hence $H_c^q(M, \mathbb{Z}) \rightarrow H^q(M, \mathbb{Z})$ is always the zero map. Now the exact sequence above shows the following.

$$H_\infty^q(M, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } q \in \{0, n-1, n, 2n-1\}, \\ 0 & \text{else.} \end{cases}$$

We consider for a positive C

$$\begin{aligned} M(C) &= \{(x, y) \in M; \sum y_i^2 = C\}, \\ M(> C) &= \{(x, y) \in M; \sum y_i^2 > C\}. \end{aligned}$$

All these sets are homotopically equivalent. Moreover, the sets $M(> C)$ define a fundamental system of open neighborhoods of ∞ . This shows

$$H^q(M(C), \mathbb{Z}) = H_\infty^q(M, \mathbb{Z}).$$

There is another way to read this result. We consider the quadric

$$Q = \left\{ z \in \mathbb{C}^{n+1}, \sum z_i^2 = 0 \right\}.$$

We want to determine the cohomology of $Q - \{0\}$. This is homotopically equivalent to

$$Q(C) = \left\{ z \in Q, \sum |z_i|^2 = C \right\}.$$

There is a topological map

$$M(1) \longrightarrow Q(2), \quad (x, y) \longmapsto z := x + iy.$$

We obtain the following result.

2.1 Lemma. *Let $Q \subset \mathbb{C}^{n+1}$ be the quadric defined by $\sum z_i^2 = 0$. Then*

$$H^q(Q - \{0\}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } q \in \{0, n-1, n, 2n-1\}, \\ 0 & \text{else.} \end{cases}$$

We need an analytic version $\text{Cl}_{\text{an}}(X)$ of the divisor class group for an irreducible normal complex space X . An analytic divisor is a formal linear combination $D = \sum_Y n_Y Y$, $n_Y \in \mathbb{Z}$, of irreducible closed complex subspaces of codimension 1 such that for each compact subset $K \subset X$ there exist only finitely many Y with $Y \cap K \neq \emptyset$ and $n_Y \neq 0$. Since X is normal, each non-zero meromorphic function f defines a divisor (f) , called a principal divisor. The divisor class group is the factor group of all divisors and the subgroup of all principal divisors. If $S \subset X$ is a closed analytic subspace of codimension ≥ 2 then $\text{Cl}_{\text{an}}(X) = \text{Cl}_{\text{an}}(X - S)$. If X is a projective variety then $\text{Cl}_{\text{an}}(X)$ and the algebraic divisor class group $\text{Cl}(X)$ are naturally isomorphic.

We need a local variant of the analytic divisor class group. Let X be a normal complex space and let $a \in X$. We set

$$\text{Cl}_{\text{an}}(X, a) := \lim_U \text{Cl}_{\text{an}}(U),$$

where U runs through all connected open neighborhoods U of a . It is of course sufficient that U runs through a fundamental system of neighborhoods.

2.2 Lemma. *Let*

$$Q := \{z \in \mathbb{C}^4; \quad z_1 z_4 = z_2 z_3\}.$$

We have

$$\mathrm{Cl}_{\mathrm{an}}(Q, 0) \cong \mathbb{Z}.$$

A generator can be given by the divisor which is defined by $z_1 = z_2 = 0$. Its negative can be defined by $z_1 = z_3 = 0$. The blow ups of these two divisors give the two small resolutions.

Proof. Since Q is a Cohen-Macaulay variety, we have that the cohomology $H_{\{0\}}^q(Q_{\mathrm{an}}, \mathcal{O}_{Q_{\mathrm{an}}})$ with support in the origin vanishes for $q < 3$. This implies

$$H^1(Q_{\mathrm{an}} - \{0\}, \mathcal{O}) = 0.$$

Here \mathcal{O} denotes the sheaf of analytic functions on $Q_{\mathrm{an}} - \{0\}$. From the exponential sequence we get

$$0 \longrightarrow H^1(Q_{\mathrm{an}} - \{0\}, \mathcal{O}^*) \longrightarrow H^2(Q_{\mathrm{an}} - \{0\}, \mathbb{Z}).$$

From Lemma 2.1 we know that $H^2(Q_{\mathrm{an}} - \{0\}) \cong \mathbb{Z}$. One can check that the line bundle related to the divisor $z_1 = z_2 = 0$ goes to a generator. This means that we get an isomorphism

$$H^1(Q_{\mathrm{an}} - \{0\}, \mathcal{O}^*) \xrightarrow{\sim} \mathbb{Z}.$$

There is natural injective map

$$\mathrm{Cl}_{\mathrm{an}}(Q - \{0\}) \longrightarrow H^1(Q_{\mathrm{an}} - \{0\}, \mathcal{O}^*).$$

Since $H^1(Q_{\mathrm{an}} - \{0\}, \mathcal{O}^*)$ is generated by the image of the divisor $z_1 = z_2 = 0$ we obtain that it is an isomorphism. We can repeat the whole consideration for

$$\left\{ z \in Q; \quad \sum |z_i|^2 < \varepsilon \right\}$$

instead of Q . This finishes the proof of Lemma 2.2. □

3. The modular approach

It is necessary for us to understand the map $\text{Cl}(\mathcal{X}) \rightarrow \text{Cl}(\mathcal{X}, a)$ for the nodes of \mathcal{X} . For this we have to use the realization of \mathcal{X} as Siegel threefold and the nodes as certain 0-dimensional cusps.

Following van Geemen and Nygaard, we described in [FS2] a certain congruence subgroup $\Gamma' \subset \text{Sp}(2, \mathbb{Z})$ such that \mathcal{X} is biholomorphic equivalent to the Satake compactification \mathbb{H}_2/Γ' of \mathbb{H}_2/Γ' , namely

$$\Gamma' = \{M \in \Gamma_2[2, 4] \cap \Gamma_{2,0,\vartheta}[4]; \quad \det D \equiv \pm 1 \pmod{8}\}.$$

For the notations we refer to [FS2].

The biholomorphic map $\mathcal{X} \cong \overline{\mathbb{H}_2/\Gamma'}$ is given by the map that assigns the variables $Y_0, \dots, Y_3, X_0, \dots, X_3$ in this ordering to the theta functions

$$\begin{aligned} \vartheta \begin{bmatrix} 00 \\ 00 \end{bmatrix}(Z), \quad \vartheta \begin{bmatrix} 00 \\ 10 \end{bmatrix}(Z), \quad \vartheta \begin{bmatrix} 00 \\ 01 \end{bmatrix}(Z), \quad \vartheta \begin{bmatrix} 00 \\ 11 \end{bmatrix}(Z), \\ \vartheta \begin{bmatrix} 00 \\ 00 \end{bmatrix}(2Z), \quad \vartheta \begin{bmatrix} 10 \\ 00 \end{bmatrix}(2Z), \quad \vartheta \begin{bmatrix} 01 \\ 00 \end{bmatrix}(2Z), \quad \vartheta \begin{bmatrix} 11 \\ 00 \end{bmatrix}(2Z). \end{aligned}$$

The nodes correspond to certain zero dimensional boundary component.

3.1 Lemma. *The standard node corresponds to the boundary point*

$$\begin{pmatrix} 0 & 0 \\ 0 & i\infty \end{pmatrix} := \lim_{t \rightarrow +\infty} \begin{pmatrix} i/t & 0 \\ 0 & it \end{pmatrix}.$$

Proof. We can use the embedded involution

$$N := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

to transform this boundary point to the standard boundary point

$$\begin{pmatrix} i\infty & 0 \\ 0 & i\infty \end{pmatrix} := \lim_{t \rightarrow +\infty} \begin{pmatrix} it & 0 \\ 0 & it \end{pmatrix}.$$

We have to transform the modular forms

$$\begin{aligned} Y_0 &= \vartheta \begin{bmatrix} 00 \\ 00 \end{bmatrix}(Z), \quad Y_1 = \vartheta \begin{bmatrix} 00 \\ 10 \end{bmatrix}(Z), \quad Y_2 = \vartheta \begin{bmatrix} 00 \\ 01 \end{bmatrix}(Z), \quad Y_3 = \vartheta \begin{bmatrix} 00 \\ 11 \end{bmatrix}(Z), \\ X_0 &= \vartheta \begin{bmatrix} 00 \\ 00 \end{bmatrix}(2Z), \quad X_1 = \vartheta \begin{bmatrix} 10 \\ 00 \end{bmatrix}(2Z), \quad X_2 = \vartheta \begin{bmatrix} 01 \\ 00 \end{bmatrix}(2Z), \quad X_3 = \vartheta \begin{bmatrix} 11 \\ 00 \end{bmatrix}(2Z) \end{aligned}$$

by means of the transformation

$$f|N(Z) = \det(CZ + D)^{-1} f(NZ).$$

Standard theta transformation formulas show that the transformed forms up to a joint constant factor are

$$Y'_0 := \vartheta \begin{bmatrix} 00 \\ 00 \end{bmatrix}(Z), \quad Y'_1 := \vartheta \begin{bmatrix} 10 \\ 00 \end{bmatrix}(Z), \quad Y'_2 := \vartheta \begin{bmatrix} 00 \\ 01 \end{bmatrix}(Z), \quad Y'_3 := \vartheta \begin{bmatrix} 10 \\ 01 \end{bmatrix}(Z)$$

and

$$\begin{aligned} X'_0 &= \frac{1}{\sqrt{2}}(\vartheta \begin{bmatrix} 00 \\ 00 \end{bmatrix}(2Z) + \vartheta \begin{bmatrix} 10 \\ 00 \end{bmatrix}(2Z)), & X'_1 &= \frac{1}{\sqrt{2}}(\vartheta \begin{bmatrix} 00 \\ 00 \end{bmatrix}(2Z) - \vartheta \begin{bmatrix} 10 \\ 00 \end{bmatrix}(2Z)) \\ X'_2 &= \frac{1}{\sqrt{2}}(\vartheta \begin{bmatrix} 01 \\ 00 \end{bmatrix}(2Z) + \vartheta \begin{bmatrix} 11 \\ 00 \end{bmatrix}(2Z)), & X'_3 &= \frac{1}{\sqrt{2}}(\vartheta \begin{bmatrix} 01 \\ 00 \end{bmatrix}(2Z) - \vartheta \begin{bmatrix} 11 \\ 00 \end{bmatrix}(2Z)). \end{aligned}$$

The projective coordinates of the cusp $\begin{pmatrix} 0 & 0 \\ 0 & i\infty \end{pmatrix}$ with respect to the variables Y_0, \dots, X_3 are the same as the coordinates of the transformed cusp $\begin{pmatrix} i\infty & 0 \\ 0 & i\infty \end{pmatrix}$ with respect to the new variables Y'_0, \dots, X'_3 . Using the formula

$$\lim_{t \rightarrow \infty} \vartheta \begin{bmatrix} a_1 a_2 \\ b_1 b_2 \end{bmatrix} \begin{pmatrix} it & 0 \\ 0 & it \end{pmatrix} = \begin{cases} 1 & \text{if } a_1 = a_2 = 0, \\ 0 & \text{else} \end{cases}$$

we get for the value $[1, 0, 1, 0, 1, 1/\sqrt{2}, 1/\sqrt{2}, 0, 0]$. This is the standard node as claimed in 3.1. \square

We denote the conjugated group of Γ' by

$$\Gamma'' := N\Gamma'N^{-1}.$$

The transformed forms Y'_0, \dots, X'_3 are generators of the ring of modular forms of Γ'' . Of course they satisfy the same relations as the Y_0, \dots, X_3 . One gets

$$\begin{aligned} \Gamma'' = \{M \in \Gamma_2[2, 4]; \quad & a_{12} \equiv d_{21} \equiv 0 \pmod{4}, \quad b_{11} \equiv c_{22} \equiv 0 \pmod{8}, \\ & a_{11}d_{22} - b_{12}c_{21} \equiv \pm 1 \pmod{8}\}. \end{aligned}$$

The Siegel-parabolic subgroup \mathcal{P} of Γ'' (defined by $C = 0$) consists of all

$$\begin{pmatrix} E & T \\ 0 & E \end{pmatrix} \begin{pmatrix} {}^tU & 0 \\ 0 & U^{-1} \end{pmatrix}, \quad U \in \mathcal{U}, \quad T \in \mathcal{T}.$$

Here \mathcal{T} denotes the set of all integral matrices $T = \begin{pmatrix} t_0 & t_1 \\ t_1 & t_2 \end{pmatrix}$ that satisfy

$$t_0 \equiv 0 \pmod{8}, \quad t_1 \equiv 0 \pmod{2}, \quad t_2 \equiv 0 \pmod{4},$$

and \mathcal{U} is the subgroup of $\mathrm{GL}(2, \mathbb{Z})$ defined by the congruences

$$b \equiv 0 \pmod{2}, \quad c \equiv 0 \pmod{4}, \quad U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This group contains non-trivial elements of finite order, for example the diagonal matrix with entries $1, -1$.

So we have proved the following result.

3.2 Lemma. *There is a biholomorphic map between \mathcal{X} and $\overline{\mathbb{H}_2/\Gamma''}$. It is defined through the correspondence $X_i \leftrightarrow X'_i$, $Y_i \leftrightarrow Y'_i$. The standard node corresponds the standard cusp $\begin{pmatrix} i\infty & 0 \\ 0 & i\infty \end{pmatrix}$. The corresponding parabolic group is the group \mathcal{P} described above.*

4. 0-dimensional cusps

We consider the Siegel parabolic group that consists of all integral symplectic matrices of genus two

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad C = 0.$$

Let \mathcal{P} be subgroup of finite index. For simplicity we assume that \mathcal{P} splits. This means the following. If M is in \mathcal{P} then the matrix

$$M = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

is in \mathcal{P} too. Let $\mathcal{U} \subset \mathrm{GL}(n, \mathbb{Z})$ be the subgroup of all U such that

$$\begin{pmatrix} {}^tU & 0 \\ 0 & U^{-1} \end{pmatrix} \in \mathcal{P},$$

and let \mathcal{T} be the set of all integral symmetric matrices T such that

$$\begin{pmatrix} E & T \\ 0 & E \end{pmatrix} \in \mathcal{P}.$$

Then \mathcal{P} consists of all

$$\begin{pmatrix} E & T \\ 0 & E \end{pmatrix} \begin{pmatrix} {}^tU & 0 \\ 0 & U^{-1} \end{pmatrix}, \quad U \in \mathcal{U}, \quad T \in \mathcal{T}.$$

The group \mathcal{U} acts on \mathcal{T} . For $U \in \mathcal{U}$ and $T \in \mathcal{T}$ we have $T[U] := {}^tUTU \in \mathcal{T}$. The group \mathcal{U} also acts on the dual lattice \mathcal{T}^* that consists of all symmetric rational matrices H such that $\sigma(TH) \in \mathbb{Z}$ for all $T \in \mathcal{T}$. This action is given by $T[{}^tU]$. Here σ denotes the trace.

Let Y be a symmetric positive definite matrix. Its minimum $m(Y)$ is defined to be the minimum value of all $Y[g]$ where g runs through all non zero integral columns. This is a continuous function on the space of all positive definite symmetric matrices. For $C \geq 0$ we denote by $\mathbb{H}_2(C)$ the set of all symmetric complex matrices Z with positive definite imaginary part Y such that $m(Y) >$

C . This is an open subset of the set of all symmetric matrices. The case $C = 0$ is the Siegel upper half-plane \mathbb{H}_2 . The group \mathcal{P} acts on $\mathbb{H}_2(C)$ through

$$\begin{pmatrix} E & T \\ 0 & E \end{pmatrix} \begin{pmatrix} {}^tU & 0 \\ 0 & U^{-1} \end{pmatrix} (Z) = Z[U] + T.$$

The quotient

$$U_C = \mathbb{H}_2(C)/\mathcal{P}$$

is a normal complex space.

We are interested in the group

$$\mathrm{Cl}(\mathcal{P}) := \lim_{C \rightarrow \infty} \mathrm{Cl}(U_C).$$

This group may be very big and we are only interested in a small part of it, the Heegner part:

Let S be a fixed matrix in \mathcal{T}^* with negative determinant. For any real number d ,

$$\{Z \in \mathbb{H}_2; \quad \sigma(ZS) = d\}$$

is a non-empty set of codimension 1. We denote by $H(S, d)$ the set of all Z with

$$\sigma(ZS[U]) \equiv d \pmod{1} \quad \text{for some } U \in \mathcal{U}.$$

This can be considered as a \mathcal{P} -invariant divisor where the multiplicities are taken to be 1. The matrix S is called primitive if it can not be written in the form $S = tS_1$ where $t > 1$ is a natural number different from 1 and $S_1 \in \mathcal{T}^*$. Then the equation $\sigma(ST) = 1$ has a solution $T \in \mathcal{T}$. Hence, in the primitive case, $H(S, d)$ is the \mathcal{P} -orbit of the divisor $\sigma(ZS) = d$. In the general case, it is a finite union of such \mathcal{P} -orbits. This follows from the trivial formula

$$H(NS, d) = \bigcup_{\nu=1}^N H\left(S, \frac{d + \nu - 1}{N}\right) \quad (S \text{ primitive, } N \in \mathbb{N}).$$

We denote the subgroup of $\mathrm{Cl}(U_C)$ spanned by these divisors by $\mathrm{Cl}_{\mathrm{Heeg}}(U_C)$. This part can be described by local Borcherds products. Local Borcherds products have been treated in the literature in different contexts. In [BF], the local divisor class group of a generic point of a one-dimensional boundary component has been treated, even more general in the context of the group $\mathrm{O}(2, n)$. Another case that has been treated by Bruinier are the cusps of Hilbert modular surfaces in [BZ]. Here we have to consider the 0-dimensional cusps of Siegel threefolds.

The convergence of the local Borcherds products simply will rest on the following result, in which we use the notation $e(Z) = \exp 2\pi i \sigma(Z)$.

4.1 Lemma. *Let $n > 0$ be a natural number. The series*

$$\sum_H e(-HY), \quad Y \text{ symmetric positive definite,}$$

converges. Here H runs through all integral matrices with determinant $\det H = -n$ and such that $\sigma(H) \geq 0$.

The proof is left to the reader. \square

We want to construct a holomorphic function with divisor $H(S, d)$. This can be done through the local Borchers product

$$B(Z) = \prod_H (e^{2\pi i \epsilon(H)(\sigma(HZ) - d)} - 1).$$

Here H runs through all matrices of the form $H = S[{}^tU]$ with $U \in \mathcal{U}$ and

$$\epsilon(H) = \begin{cases} +1 & \text{if } \sigma(H) \geq 0, \\ -1 & \text{else.} \end{cases}$$

From Lemma 4.1 follows that this product converges in \mathbb{H}_2 and defines an analytic function there. The function B is periodic with respect to \mathcal{T} .

4.2 Lemma. *Let $B(Z)$ be the local Borchers product with zero set $H(S, d)$. Assume that $H(S, d)$ is in the ramification of the map $\mathbb{H}_2(C)/\mathcal{T} \rightarrow \mathbb{H}_2(C)/\mathcal{P}$. Then the multiplicity of the zeros of $B(Z)$ is two. This is also the ramification order.*

Proof. Assume that the set $\sigma(SZ) = d$ is in the ramification. This means that there exists a substitution $Z \mapsto Z[U] + H$ in \mathcal{P} that fixes it. This transformation must be of finite order, and hence U is of finite order. The set

$$\{Y > 0; \quad \sigma(SY) = 0\}$$

is not empty and two-dimensional. The equation $Y[U] = Y$ holds on this set. We claim that $\det U = -1$. This follows from the fact that an element from $\mathrm{SL}(2, \mathbb{Z})$ has at most one fixed point with respect to the standard action of $\mathrm{SL}(2, \mathbb{Z})$ on the upper half plane. The same argument shows $U^2 = \pm E$. The minus sign is not possible, since otherwise the two eigenvalues of U would be equal (both $\pm i$) and then U would be a multiple of the unit matrix. The known list of the elements of finite order in $\mathrm{GL}(2, \mathbb{Z})$ shows that each of the matrices U is conjugate to one of the following two:

$$U_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In the first case the solutions of $\sigma(Y[U_1]) = Y$ are the diagonal matrices. The corresponding matrix S must be a constant multiple of U_2 . Analogously, the matrix S that corresponds to U_2 is a constant multiple of U_1 . In both cases we have $S[{}^tU] = -S$. Hence this holds in general. We also see that the ramification order is two.

We know

$$\sigma(SZ) = d \implies \sigma(S(Z[U] + H) = d.$$

Hence we get

$$\sigma(SZ) = d \implies \sigma(SZ) = \sigma(SH) - d.$$

This gives $2d = \sigma(SH) \in \mathbb{Z}$. Since in the Borchers product besides $H = S$ also $H = -S = S[{}^tU]$ occurs, we obtain that the set $\sigma(SZ) = d$ is a at least double zero. It is easy to see that the multiplicity is really 2. \square

For $U \in \mathcal{U}$ we have that

$$J(U, Z) := B(Z[U])/B(Z)$$

is a cocycle,

$$J(UV, Z) = J(V, Z[U])J(U, Z).$$

(As in group cohomology usual, we consider the action of \mathcal{U} from the left. This is given by $(U, f) \mapsto g$ where $g(Z) = Z[U]$.) So $J(U, Z)$ represents an element of

$$H^1(\mathcal{U}, \mathcal{O}^*(\mathbb{H}_2(C)/\mathcal{T})).$$

We denote the part of this group that is generated by the $J(U, Z)$ above by

$$H_{\text{Heeg}}^1(\mathcal{U}, \mathcal{O}^*(\mathbb{H}_2(C)/\mathcal{T})).$$

Let $B(Z)$ be a local Borchers product. Its zero divisor, considered on $\mathbb{H}_2(C)$, is invariant under \mathcal{P} . Hence it induces a divisor on U_C . By definition, the multiplicity of a component is the multiplicity of B considered on $\mathbb{H}_2(C)$ divided by the ramification order (which is 1 or 2). By Lemma 4.2 this quotient is an integer. Hence we obtain the following result.

4.3 Lemma. *There is a homomorphism*

$$H_{\text{Heeg}}^1(\mathcal{U}, \mathcal{O}^*(\mathbb{H}_2(C)/\mathcal{T})) \longrightarrow \text{Cl}(U_C)$$

that attaches to the class of the cocycle $B(Z[U])/B(Z)$ the induced divisor on U_C with multiplicities as described above.

An automorphy factor coming from a local Borchers product is of a very simple form because of the following lemma.

4.4 Lemma. *Let $U \in \mathrm{GL}(2, \mathbb{Z})$ and $n > 0$ a natural number. There exist only finitely many symmetric integral T with the properties*

$$\det T = -n, \quad \epsilon(T) \neq \epsilon(T[U]).$$

The proof is left to the reader. \square

The lemma implies that the automorphy factor $J(U, Z) = B(Z[U])/B(Z)$ is of a very simple form. We have to consider quotients of the type

$$\frac{e^x - 1}{e^{-x} - 1} = -e^x.$$

Hence we see: The cocycle related to a local Borchers product is of the form

$$J(U, Z) = C_U e(ZH_U),$$

where C_U is a constant of absolute value 1 and $H_U \in \mathcal{T}^*$. They are given as follows:

4.5 Lemma. *The cocycle of the local Borchers product*

$$B(Z) = \prod_H (e^{2\pi i \epsilon(H)(\sigma(HZ) - d)} - 1)$$

associated to $H(S, d)$ is of the form

$$J(U, Z) := B(Z[U])/B(Z) = C_U e^{2\pi i \sigma(ZH_U)}.$$

If \mathcal{H} denotes the (finite) set of all $H = S[{}^tV]$, $V \in \mathcal{U}$, such that $\epsilon(H) \neq \epsilon(H[{}^tU^{-1}])$, then

$$H_U = - \sum_{H \in \mathcal{H}} \epsilon(H) H,$$

and

$$C_U = (-1)^{\#\mathcal{H}} \exp\left(2\pi i d \sum_{H \in \mathcal{H}} \epsilon(H)\right).$$

We call the \mathcal{U} -module of the functions

$$C \cdot e(ZH); \quad |C| = 1, \quad H \in \mathcal{T}^*,$$

by \mathcal{E} . Then our cocycles are in the image of the natural map

$$H^1(\mathcal{U}, \mathcal{E}) \longrightarrow H^1(\mathcal{U}, \mathcal{O}^*(\mathbb{H}_2(C)/\mathcal{T})).$$

Remark. We also can consider the submodule $\mathcal{E}_0 \subset \mathcal{E}$ of all elements with $C = 1$. Then $\mathcal{E}_0 \otimes_{\mathbb{Z}} \mathbb{R}$ is isomorphic to $\mathrm{Sym}^2(\mathbb{R}^2)$. Hence there is a link to the Eichler cohomology group $H^1(\mathcal{U}, \mathrm{Sym}^2(\mathbb{R}^2))$ and from there are relations to elliptic cusp forms of weight 4.

We denote the part of $H^1(\mathcal{U}, \mathcal{E})$ coming from local Borchers products by $H_{\mathrm{Heeg}}^1(\mathcal{U}, \mathcal{E})$. We have proved the following result.

4.6 Lemma. *There is a natural homomorphism*

$$H_{\text{Heeg}}^1(\mathcal{U}, \mathcal{E}) \longrightarrow \text{Cl}(U_C).$$

5. A very particular case

We take for \mathcal{P} the parabolic group described in Sect. 3.

5.1 Lemma. *The group \mathcal{U} can be generated by the matrices*

$$V_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}.$$

We skip the proof and simply mention that the group $\mathcal{U} \cap \text{SL}(2, \mathbb{Z})$ is conjugate to the group $\Gamma_0[8]$. One can use the program MAGMA [BMP] to get generators. \square

5.2 Theorem. *In the special case of \mathcal{P} above, we have*

$$\lim_{C \rightarrow \infty} \text{Cl}(U_C) \cong \mathbb{Z}.$$

Proof. Recall (Sect. 2) that there exists a subgroup $\Gamma'' \subset \text{Sp}(2, \mathbb{Z})$ of finite index such that \mathcal{P} is just the parabolic subgroup defined by $C = 0$. We use some details of the Satake compactification $X_{\Gamma}'' = \mathbb{H}_2^*/\Gamma''$ as described in [Fr] for example. For C large enough the natural map

$$\mathbb{H}_2(C)/\mathcal{P} \longrightarrow \mathbb{H}_2/\Gamma''$$

is an open embedding. Hence we can consider then U_C as a subset of \mathbb{H}_2/Γ'' . There exists a fundamental system of open neighborhoods \hat{U}_C of the standard zero dimensional boundary component ∞ in $X_{\Gamma''}$ such that

$$\hat{U}_C \cap (\mathbb{H}_2/\Gamma) = U_C \quad (C \gg 0).$$

The complement of U_C in \hat{U}_C is a curve. From [FS2] we know that the germ (X_{Γ}'', ∞) is isomorphic to the germ of a quadric at the origin. In [FS2] we have taken the quadric Q defined by $z_1 z_4 = z_2 z_3$. Now we go back to $\hat{U}(C)$. The groups $\text{Cl}(U_C)$ and $\text{Cl}(\hat{U}_C - \{\infty\})$ agree, since divisors can be extended over codimension ≥ 2 . Now we can apply Lemma 2.2 to complete the proof. \square

We are interested in some very simple divisors, namely those which correspond to the divisors D_i^{\pm} in Sect. 2. Using the coordinates described in Lemma 3.2, we have the following result.

5.3 Lemma. *The equations of the transformed divisors D_i^\pm in the model $\overline{\mathbb{H}_2/\Gamma''}$ are (set theoretically):*

$$\begin{aligned} D_1^\pm : \quad & \vartheta \begin{bmatrix} 11 \\ 00 \end{bmatrix} (2Z) = \vartheta \begin{bmatrix} 10 \\ 00 \end{bmatrix} (Z) \pm \vartheta \begin{bmatrix} 10 \\ 01 \end{bmatrix} (Z) = 0, \\ D_2^\pm : \quad & \vartheta \begin{bmatrix} 01 \\ 00 \end{bmatrix} (Z)^2 = \vartheta \begin{bmatrix} 00 \\ 00 \end{bmatrix} (Z) \vartheta \begin{bmatrix} 10 \\ 00 \end{bmatrix} (Z) \pm \vartheta \begin{bmatrix} 00 \\ 01 \end{bmatrix} (Z) \vartheta \begin{bmatrix} 10 \\ 01 \end{bmatrix} (Z) = 0, \\ D_3^\pm : \quad & \vartheta \begin{bmatrix} 10 \\ 00 \end{bmatrix} (2Z) - \vartheta \begin{bmatrix} 01 \\ 00 \end{bmatrix} (2Z) = \\ & \vartheta \begin{bmatrix} 10 \\ 00 \end{bmatrix} (Z) \vartheta \begin{bmatrix} 00 \\ 10 \end{bmatrix} (Z/2) \pm \vartheta \begin{bmatrix} 00 \\ 01 \end{bmatrix} (Z) \vartheta \begin{bmatrix} 10 \\ 01 \end{bmatrix} (Z) = 0. \end{aligned}$$

The proof is a straightforward calculation. We only mention that the theta relations

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (Z)^2 = \sum_x (-1)^{t_{xb}} \vartheta \begin{bmatrix} a+x \\ 0 \end{bmatrix} (2Z) \vartheta \begin{bmatrix} x \\ 0 \end{bmatrix} (2Z).$$

$$\vartheta \begin{bmatrix} 00 \\ 10 \end{bmatrix} (Z/2) = \vartheta \begin{bmatrix} 00 \\ 00 \end{bmatrix} (2Z) - \vartheta \begin{bmatrix} 10 \\ 00 \end{bmatrix} (2Z) + \vartheta \begin{bmatrix} 01 \\ 00 \end{bmatrix} (2Z) - \vartheta \begin{bmatrix} 11 \\ 00 \end{bmatrix} (2Z)$$

have been used. □

We also can give the equations in the Siegel upper half plane.

5.4 Lemma. *Close to the standard boundary point, the divisors D_1^+, \dots, D_3^- can be defined as images of the following sets.*

$$\begin{aligned} D_1^+ : \quad & 2z_1 + 4z_2 = 1, & D_1^- : \quad & 2z_1 = 1, \\ D_2^+ : \quad & z_0 + 3z_1 + 2z_2 = 1, & D_2^- : \quad & z_0 + z_1 = 1, \\ D_3^+ : \quad & z_2 = z_0 + 2, & D_3^- : \quad & z_2 = z_0. \end{aligned}$$

Proof. We start with the divisor D_1^- : We restrict the series

$$\begin{aligned} & \vartheta \begin{bmatrix} a_1 a_2 \\ b_1 b_2 \end{bmatrix} (2Z) = \\ & \sum (-1)^{b_1 g_1 + b_2 g_2} e^{2\pi i (z_0 (g_1 + a_1/2)^2 + 2z_1 (g_1 + a_1/2)(g_2 + a_2/2) + z_2 (g_2 + a_2/2)^2)} \end{aligned}$$

to $2z_1 = 1$. Since $2g_1 g_2$ is even we get

$$e^{\pi i a_1 a_2 / 2} \vartheta \begin{bmatrix} a_1 \\ b_1 + a_2 \end{bmatrix} (z_0) \cdot \vartheta \begin{bmatrix} a_2 \\ b_2 + a_1 \end{bmatrix} (z_2).$$

Since $\vartheta[\mathbf{m}](z)$ vanishes for the odd characteristic $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, we get that $\vartheta \begin{bmatrix} 11 \\ 00 \end{bmatrix} (2Z)$ vanishes along $2z_1 = 1$.

Next we restrict $\vartheta \begin{bmatrix} 10 \\ 00 \end{bmatrix} (Z)$ to $2z_1 = 1$. The result is

$$\sum e^{\pi i (z_0 (g_1 + 1/2)^2 + (g_1 + 1/2)g_2 + z_2 g_2^2)}.$$

we replace g_1 by $-g_1 - 1$. Then we use that

$$(-g_1 - 1/2)g_2 \equiv (g_1 + 1/2)g_2 + g_2 \pmod{2}.$$

This shows that $\vartheta \begin{bmatrix} 10 \\ 00 \end{bmatrix} (Z)$ and $\vartheta \begin{bmatrix} 10 \\ 01 \end{bmatrix} (Z)$ agree on $2z_1 = 1$. This settles D_1^- .

Next we treat D_1^+ : We use the unimodular transformation $Z \mapsto Z \begin{bmatrix} 10 \\ 21 \end{bmatrix}$. This transforms the set $2z_1 + 4z_2 = 1$ to the set $2z_1 = 1$. The form $\vartheta \begin{bmatrix} 11 \\ 00 \end{bmatrix} (2Z)$ is invariant under this transformation. Hence it vanishes also on $2z_1 + 4z_2 = 1$. Similarly $\vartheta \begin{bmatrix} 10 \\ 00 \end{bmatrix} (Z)$ is invariant, but $\vartheta \begin{bmatrix} 10 \\ 01 \end{bmatrix} (Z)$ changes its sign. Hence D_1^+ has been reduced to D_1^- .

Next we treat D_2^- . We use the transformation $Z \mapsto Z \begin{bmatrix} 11 \\ 01 \end{bmatrix}$. It transforms the set $z_0 + z_1 = 1$ to the set $z_1 = 1$ and $\vartheta \begin{bmatrix} 01 \\ 00 \end{bmatrix} (Z)^2$ is transformed to $\vartheta \begin{bmatrix} 11 \\ 00 \end{bmatrix} (Z)^2$. We have already seen that this vanishes along $z_1 = 1$. The series

$$\vartheta \begin{bmatrix} 00 \\ 00 \end{bmatrix} (Z), \quad \vartheta \begin{bmatrix} 10 \\ 00 \end{bmatrix} (Z), \quad \vartheta \begin{bmatrix} 00 \\ 01 \end{bmatrix} (Z), \quad \vartheta \begin{bmatrix} 10 \\ 01 \end{bmatrix} (Z)$$

are invariant under the transformation. For the restriction to $z_1 = 1$ we use that $\vartheta \begin{bmatrix} a_1 a_2 \\ b_1 b_2 \end{bmatrix} (Z)$ goes (see above) to

$$e^{\pi i a_1 a_2 / 2} \vartheta \begin{bmatrix} a_1 \\ b_1 + a_2 \end{bmatrix} (z_0) \cdot \vartheta \begin{bmatrix} a_2 \\ b_2 + a_1 \end{bmatrix} (z_2).$$

Hence

$$\vartheta \begin{bmatrix} 00 \\ 00 \end{bmatrix} (Z) \vartheta \begin{bmatrix} 10 \\ 00 \end{bmatrix} (Z) - \vartheta \begin{bmatrix} 00 \\ 01 \end{bmatrix} (Z) \vartheta \begin{bmatrix} 10 \\ 01 \end{bmatrix} (Z)$$

goes to

$$\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z_0) \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z_2) \vartheta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (z_0) \vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (z_2) - \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z_0) \vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (z_2) \vartheta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (z_0) \vartheta \begin{bmatrix} 0 \\ 2 \end{bmatrix} (z_2)$$

which is zero. This settles D_2^- .

To treat D_2^+ we use the transformation $Z \mapsto Z \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$. It transforms the set $z_0 + z_1 = 1$ to the set $z_0 + 3z_1 + 2z_2 = 1$. It leaves $\vartheta \begin{bmatrix} 01 \\ 00 \end{bmatrix} (Z)^2$ invariant. Hence this function vanishes along $z_0 + 3z_1 + 2z_2 = 1$. We have seen already that it changes the sign of $\vartheta \begin{bmatrix} 10 \\ 01 \end{bmatrix} (Z)$. This shows the transformation of the expression $\vartheta \begin{bmatrix} 00 \\ 00 \end{bmatrix} (Z) \vartheta \begin{bmatrix} 10 \\ 00 \end{bmatrix} (Z) - \vartheta \begin{bmatrix} 00 \\ 01 \end{bmatrix} (Z) \vartheta \begin{bmatrix} 10 \\ 01 \end{bmatrix} (Z)$ just causes a change of the minus sign to a plus sign. Now D_2^\pm is settled.

We treat D_3^- . First one sees that the transformation $Z \mapsto Z \begin{bmatrix} 01 \\ 10 \end{bmatrix}$ interchanges $\vartheta \begin{bmatrix} 10 \\ 00 \end{bmatrix} (2Z)$ and $\vartheta \begin{bmatrix} 01 \\ 00 \end{bmatrix} (2Z)$. Since this transformation is the identity on $z_0 = z_2$, the difference vanishes on this set.

We still have to investigate the four series

$$A: \quad \vartheta \begin{bmatrix} 10 \\ 00 \end{bmatrix} (Z), \quad B: \quad \vartheta \begin{bmatrix} 00 \\ 10 \end{bmatrix} (Z/2),$$

$$C : \vartheta \begin{bmatrix} 00 \\ 01 \end{bmatrix}(Z), \quad D : \vartheta \begin{bmatrix} 10 \\ 01 \end{bmatrix}(Z).$$

The claim is that $AB = CD$ on $z_0 = z_2$. We use the transformation $Z \mapsto Z \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Then the set $z_0 = z_2$ transforms to $z_1 = 0$. We will use the formula

$$Z \left[\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right] = z_0(g_1 + g_2)^2 + z_2(g_1 - g_2)^2 \quad \text{for } z_1 = 0.$$

If g_1, g_2 runs through all integers than $h_1 = g_1 + g_2$, $h_2 = g_1 - g_2$ runs through all pairs of integers such that $h_1 + h_2$ is even. For A we get the expression

$$\sum_{h_1+h_2 \text{ even}} e^{\pi i(z_0(h_1+1/2)^2+z_2(h_2+1/2)^2)}.$$

We divide this sum into two parts where h_1, h_2 both are even or both are odd. Both partial sum are equal and we get:

$$A : \quad 2\vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(4z_0)\vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(4z_2).$$

In a similar way we split B, C, D . The result is

$$\begin{aligned} B : & \quad \vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix}(2z_0)\vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix}(2z_2), \\ C : & \quad \vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix}(4z_0)\vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix}(4z_2), \\ D : & \quad 2e^{-\pi i/4}\vartheta \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}(4z_0)\vartheta \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}(4z_2). \end{aligned}$$

The relation $AB = CD$ means:

$$\vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(2z)\vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix}(z) = e^{-\pi i/4}\vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix}(2z)\vartheta \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}(2z).$$

This relation between theta series of one variable is left as an exercise.

The case D_3^+ is similar. □

5.5 Lemma. *The divisors D_i^- correspond (close to the standard node) to divisors of the form $H(S, d)$:*

$$\begin{aligned} D_1^- : & \quad S = 1/4 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad d = \frac{1}{4}, \\ D_2^- : & \quad S = 1/4 \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \quad d = \frac{1}{2}, \\ D_3^- : & \quad S = 1/4 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad d = 0. \end{aligned}$$

Notice that S is a primitive element of \mathcal{T}^* . We compute the value $J(U, Z)$ of the cocycle of the associated local Borchers product for some U . Recall (4.5) that

$$J(U, Z) := B(Z[U])/B(Z) = C_U e^{2\pi i \sigma(ZH_U)}.$$

The constant C_U is not important, since it is a root of unity that disappears if one takes a suitable power. Hence we only give our attention to H_U . We compute them for the generators given in Lemma 5.1. Evaluation of the formula given in Lemma 4.5 gives the following values for the $8H_{V_i}$:

$$\begin{aligned} D_1^- : & \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -4 & -2 \\ -2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -2 \\ -2 & -8 \end{pmatrix}, \quad \begin{pmatrix} -12 & -18 \\ -18 & -24 \end{pmatrix}. \\ D_2^- : & \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -4 & -2 \\ -2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \\ D_3^- : & \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -4 & -2 \\ -2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

To compare the we make use of trivial divisors $e^{\sigma(UZ^tU)}/e^{\sigma(Z)}$. For $U = V_1$ and $U = V_2$ one gets the following values

$$\begin{aligned} (V_1) : & \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 4 \\ 4 & 16 \end{pmatrix}, \quad \begin{pmatrix} 8 & 12 \\ 12 & 16 \end{pmatrix}. \\ (V_2) : & \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 4 & 6 \\ 6 & 8 \end{pmatrix}. \end{aligned}$$

From these data follows the following proposition.

5.6 Proposition. *Consider the divisors $D_1^\pm, D_2^\pm, D_3^\pm$. Their classes in $\text{Cl}(\mathcal{X}, \eta)$ are non zero and satisfy the relations $D_i^+ = -D_i^-$ and*

$$D_1^\pm = -D_2^\pm = -D_3^\pm.$$

The stabilizer of a node a acts on the local Picard group $\text{Cl}(\mathcal{X}, a)$. The subgroup of index two that fixes the rulings acts as identity. Hence we see the following result.

5.7 Remark. *By means of Proposition 5.6 and Theorem 1.4 it is possible to compute the image of each of the 188 basic divisors in $\text{Cl}(\mathcal{X}, \eta)$. Since they generate $\text{Cl}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ we have a complete description of the map*

$$\text{Cl}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \text{Cl}(\mathcal{X}, \eta) \otimes_{\mathbb{Z}} \mathbb{Q} \quad (\cong \mathbb{Q}).$$

Hyperplane sections of \mathcal{X} define line bundles and hence get trivial in the local divisor class groups at nodes. Hence the one dimensional representation is in the kernel of the map

$$\mathrm{Cl}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \bigoplus_{a \text{ node}} \mathrm{Cl}(\mathcal{X}, a) \otimes_{\mathbb{Z}} \mathbb{Q}$$

From the above explicit description one can compute that the one dimensional representation is the precise kernel. This might be a general phenomenon for Siegel threefolds (compare [BF], Theorem 5.4).

6. Projective Resolutions

If D is an effective divisor on an irreducible normal complex space X , then we associate the ideal sheaf $\mathcal{I}(D)$ of all holomorphic functions on open subsets that satisfy $(f) \geq D$ on this subset. The blow up of D , by definition, is the blow up of the ideal sheaf $\mathcal{I}(D)$. The blow up of an effective divisor depends only on its image in the divisor class group $\mathrm{Cl}(X)$.

The relation $\mathcal{I}(D)^n = \mathcal{I}(nD)$ is not true in general. But in our situation it is true.

6.1 Lemma. *Let Q be the affine quadric defined by $z_1 z_4 = z_2 z_3$ and let P be the prim divisor defined by $z_1 = z_2 = 0$. Then*

$$\mathcal{I}(nP) = \mathcal{I}(P)^n.$$

Proof. We have to show $\mathcal{I}(nP) \subset \mathcal{I}(P)^n$ for $n > 0$ since the converse inclusion is trivial. Hence we have to consider an element f of the (analytic) local ring $\mathcal{O}_{Q,0}$ that vanishes along P of order at least n . We can take this element as the restriction of a power series

$$F \in R := \mathbb{C}\{z_1, z_2, z_3, z_4\}.$$

We expand F as power series of the variables z_1, z_2 ,

$$F = \sum g_{i,j}(z_3, z_4) z_1^i z_2^j.$$

Using the relation $z_1 z_4 = z_2 z_3$ we can modify F in such a way that $g_{i,j}$ is independent of z_3 in the case $j > 0$. Now we claim that $g_{i,j} = 0$ if $i + j < n$. The proof is given by induction on $k = i + j$. The beginning of the induction

($k = 0$ is trivial. To get the idea, we treat the case $k = 1$ (assuming $n \geq 2$). We know that the restriction of

$$g_{1,0}z_1 + g_{0,2}z_2$$

to Q vanishes along $z_1 = z_2 = 0$ in at least second order. We test this in the chart $z_3 \neq 0$. The divisor is given in this chart by one equation $z_1 = 0$. The claim is that

$$g_{1,0}(z_3, z_4)z_1 + g_{0,1}(z_4)z_1z_4/z_3$$

vanishes of at least second order on $z_1 = 0$. Then

$$g_{1,0}(z_3, z_4) + g_{0,1}(z_4)z_4/z_3$$

still must vanish along $z_1 = 0$. Since it is independent of z_1 it must vanish. This gives

$$g_{0,1}(z_4)z_4 = -g_{1,0}(z_3, z_4)z_3.$$

Since the left-hand side is independent of z_3 (due to our normal form) we get $g_{0,1} = 0$ and then $g_{1,0} = 0$. \square

Let (X, a) be a three dimensional nodal singularity. The blow up of a divisor D gives a small resolution of the node if and only if D runs non-trivially into the node in the following sense: the image of D in the local divisor class group $\text{Cl}(X, a)$ is not zero. This follows easily from Lemma 6.1 in connection with the the structure theorem $\text{Cl}(X, a) \cong \mathbb{Z}$.

Let now $G \subset \bar{\mathcal{H}}$ be a subgroup. We recall that we defined at the end of the introduction two sets \mathcal{A}, \mathcal{B} of nodes. We reformulate the theorem at the end of the introduction in the following way.

6.2 Proposition. *Let $G \subset \bar{\mathcal{H}}$ be a subgroup such that $\mathcal{A} \cup \mathcal{B}$ is the set of all nodes. Assume that for each node $a \in \mathcal{B}$, $a \notin \mathcal{A}$, the map*

$$(\text{Cl}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q})^G \longrightarrow \text{Cl}(\mathcal{X}, a) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is not the zero map. Then there exists a projective resolution $\tilde{\mathcal{X}} \rightarrow \mathcal{X}$ such that the action of G extends to a group of biholomorphic transformations of $\tilde{\mathcal{X}}$ and such that $G\text{-Hilb}(\tilde{\mathcal{X}})$ is a resolution of \mathcal{X}/G in the form of a projective Calabi–Yau manifold.

Proof. First we blow up the nodes from \mathcal{A} to get a partial resolution $\mathcal{X}_1 \rightarrow \mathcal{X}$. The group G extends since the set \mathcal{A} is G -invariant. If \mathcal{X}_1 is smooth we are done. Otherwise we choose some node $a \in \mathcal{X}$ that remains singular in \mathcal{X}_1 . By assumption there exists a divisor D such that its class in $\text{Cl}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is G -invariant and that is non-trivial at a . We can assume that D is effective, since we can add to D the divisor of a homogenous polynomial in Y_0, \dots, X_3 . We can also assume that D itself (and not only its class) is G -invariant, since we

can replace G by the sum of all $g(D)$, $g \in G$. Now we want to blow up D . To be precise, we take the transform D_1 of D in \mathcal{X}_1 , and blow up D_1 to produce a partial resolution $\mathcal{X}_2 \rightarrow \mathcal{X}_1$. Of course the blow up doesn't change anything in the smooth part of \mathcal{X}_1 because there any divisor is locally principal. The resolution $\mathcal{X}_2 \rightarrow \mathcal{X}_1$ is a small resolution of certain nodes including the node a . If \mathcal{X}_2 is already smooth we are done. Otherwise we proceed in the same manner.

The constructed $\tilde{\mathcal{X}}$ has the property that for each point of the quotient $\tilde{\mathcal{X}}/G$ there exists a small (analytic) neighborhood U and a holomorphic 3-form without zeros on the regular locus of U . For the nodes $a \in \mathcal{A}$ one has to observe the following. The Calabi–Yau form gets a zero of order one along the exceptional divisor of the blow up. But, by the definition of \mathcal{A} there is an (central) element $g \in G_a$ that fixes the exceptional divisor pointwise. Due to the ramification, the Calabi–Yau form, considered on the quotient of the blow up by g , doesn't vanish along the image of the exceptional divisor.

As a consequence, all singularities of $\tilde{\mathcal{X}}/G$ are of the form \mathbb{C}^3/H , where $H \subset \mathrm{SL}(3, \mathbb{C})$, is a finite group. But then the results of [BKS] show that the G -Hilbert scheme $G\text{-Hilb}(\tilde{\mathcal{X}})$ gives a projective resolution in the form of a Calabi–Yau manifold (compare [FS1], Theorem 2.6). \square

7. Examples

1) It can happen that a group $G \subset \bar{\mathcal{H}}$ contains the conjugates of the transformation $(Y_0, -Y_1, Y_2, -Y_3, X_0, X_1, -X_2, -X_3)$. For such a group all nodes are contained in \mathcal{A} . Hence \mathcal{X}/G admits a projective Calabi–Yau resolution. These examples has been described in detail in [FS2] (see Lemma 8.1, Remark 8.3 and Corollary 8.4).

2) We start with a counter example. There exist groups $G \subset \mathcal{G}$ of order 32 that acts freely on \mathcal{X} . An example is the group generated by the following transformations.

$$\begin{aligned} &(-Y_1, -iY_0, Y_3, -iY_2, -iX_1, X_0, iX_3, X_2), \\ &(-iY_2, Y_3, Y_0, iY_1, X_3, -X_2, -iX_1, -iX_0), \\ &(-Y_0, Y_1, Y_2, -Y_3, X_0, -X_1, X_2, -X_3), \\ &(iY_0, iY_1, iY_2, iY_3, -iX_0, -iX_1, -iX_2, -iX_3), \\ &(Y_0, Y_1, -Y_2, -Y_3, X_0, X_1, -X_2, -X_3), \\ &(iY_0, iY_1, iY_2, iY_3, iX_0, iX_1, iX_2, iX_3) \end{aligned}$$

In this case all nodes are of type \mathcal{B} . But the space $(\mathrm{Cl}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q})^G$ is one dimensional (generated by a hyperplane section). The image in $\mathrm{Cl}(\mathcal{X}, a) \otimes_{\mathbb{Z}} \mathbb{Q}$

is zero for all nodes a . We claim even more. There is no projective Calabi–Yau manifold M that is birational equivalent to \mathcal{X}/G . This follows from a theorem of Kollar that states that two bimeromorphic equivalent models are related by flops ([Ko], Theorem 4.9). Hence M must be obtained from \mathcal{X}/G by a small resolution. The pull back to \mathcal{X} would give a projective G -invariant resolution. It is easy to show that this is the blow up of a G -invariant divisor.*)

Using calculator we got the following result.

7.1 Theorem. *There are 54 conjugacy classes of subgroups $G \subset \bar{\mathcal{H}}$ that act freely on \mathcal{X} . Their orders are in $\{1, 2, 4, 8, 16, 32\}$. For those of order 32 there is no projective Calabi–Yau model for \mathcal{X}/G (but a weak Calabi–Yau model). Those of order 1, 2 and 4 all admit projective Calabi–Yau models. In the case of order 8, 13 cases have a projective C–Y model and 7 cases do not have it. There is one class of order 16 with a projective Calabi–Yau model, the other 12 classes do not have it.*

The quotient of a rigid projective manifold M by a finite freely acting group is rigid. The Euler number is $e(M/G) = e(M)/\#G$. Hence we get rigid Calabi–Yau manifolds with Euler numbers 4, 8, 16, 32.

The freely acting group of order 16 with a projective Calabi–Yau model is of special interest.

7.2 Theorem. *The group G generated by the two transformations*

$$\begin{aligned} &(-Y_2, iY_3, iY_0, Y_1, -iX_3, -iX_2, X_1, -X_0), \\ &(iY_1, Y_0, -iY_3, Y_2, X_1, iX_0, X_3, -iX_2) \end{aligned}$$

has order 16 and acts freely on \mathcal{X} . The quotient \mathcal{X}/G has a resolution in the form of a rigid Calabi–Yau manifold ($h^{12} = 0$) with Euler number $e = 4$ and Picard number $h^{11} = 2$.

3) We consider the group G of order 2 that is generated by the involution

$$\sigma_1 : (Y_0, Y_1, Y_2, Y_3, -X_0, -X_1, -X_1, -X_3).$$

It generates a normal subgroup G of $\bar{\mathcal{G}}$. The involution is fixed point free. Hence all nodes are of type \mathcal{B} (and there are no ones of type \mathcal{A}). The divisor $D_2^+ (X_0X_2 + X_1X_3 = Y_0Y_1 \pm Y_2Y_3 = 0)$ is invariant under G and it is non-trivial at the standard node (see Proposition 5.6). Since G is normal we obtain by transformation for each node a divisor in that is G -invariant and non-trivial at this node. Hence we get a projective model.

*) We are very grateful to van Geemen who pointed out to us that there exists no projective model for this group G .

4) Next we consider the group of order 2 that is generated by the involution

$$\sigma_2 := (Y_0, -Y_1, -Y_2, Y_3, X_0, -X_1, -X_2, X_3).$$

This substitution has 6 conjugates in $\bar{\mathcal{H}}$, namely

$$\begin{aligned} &(Y_0, -Y_1, Y_2, -Y_3, -X_0, -X_1, X_2, X_3), (Y_0, -Y_1, -Y_2, Y_3, X_0, -X_1, -X_2, X_3), \\ &(Y_0, Y_1, -Y_2, -Y_3, -X_0, X_1, -X_2, X_3), (Y_0, -Y_1, -Y_2, Y_3, -X_0, X_1, X_2, -X_3), \\ &(Y_0, Y_1, -Y_2, -Y_3, X_0, -X_1, X_2, -X_3), (Y_0, -Y_1, Y_2, -Y_3, X_0, X_1, -X_2, -X_3). \end{aligned}$$

Again the divisor D_2^+ is invariant under them and not trivial at the standard node. So we get projectivity again.

There are 10 conjugacy classes of involutions in $\bar{\mathcal{H}}$. In [FS2] we listed them in Proposition 7.6 and computed the divisor class and Euler numbers for them. As in the case σ_1, σ_2 above one can verify in each of the cases that 6.2 applies to obtain a projective Calabi–Yau manifold again. Then we obtain

7.3 Theorem. *The following table describes fixed point sets of the involutions σ_i , $i \leq 10$, on \mathcal{X} and the Hodge numbers of a (projective) Calabi–Yau model of the quotient \mathcal{X}/σ_i .*

	fixed points	h^{11}	h^{12}
σ_1 :	empty set	16	0
σ_2 :	16 nodes	40	0
σ_3 :	4 elliptic curves	20	4
σ_4 :	empty set	16	0
σ_5 :	empty set	16	0
σ_6 :	8 conics in planes ($\cong \mathbb{P}^1$)	28	0
σ_7 :	8 lines ($\cong \mathbb{P}^1$)	28	0
σ_8 :	2 elliptic curves	18	2
σ_9 :	2 elliptic curves	18	2
σ_{10} :	4 conics in planes ($\cong \mathbb{P}^1$)	22	0

In the cases where elliptic curves are in the fixed point set, we get *non rigid Calabi–Yau manifolds*.

5) Next we consider a group of order 3.

7.4 Proposition. *There is only one conjugacy class of elements of order three in $\bar{\mathcal{H}}$. It can be represented by*

$$(Y_0, Y_3, Y_1, Y_2, X_0, X_3, X_1, X_2).$$

Its fixed point set in \mathcal{X} doesn't contain a node. It consists of one elliptic curve and 4 isolated points. The fixed point set in $\tilde{\mathcal{X}}$ is the same. The subgroup of order three leads to a projective Calabi–Yau model with Hodge numbers

$$(h^{11}, h^{12}) = (18, 2).$$

Proof (sketch). We omit the computation of the fixed points. We can extend the group to a small resolution $\tilde{\mathcal{X}}$ of \mathcal{X} . Now the Euler number easily can be computed by means of the stringy formula (compare [FS2]). The result is $e = 32$. Next we compute the Picard number $h^{1,1}$. The invariant part $\dim \mathrm{Cl}(\mathcal{X})^G$ has rank 12 as can be proved by means of the known character of the action of $\bar{\mathcal{H}}$ (see [FS2]). One can check that over isolated points there is one exceptional divisor but over the elliptic curve there are two. Hence we have 6 exceptional divisors. We obtain $h^{1,1} = 18$. \square

8. A Special class of groups

This section rests highly on computer calculations. We consider all subgroups of \mathcal{H} that are isomorphic to $(\mathbb{Z}/2\mathbb{Z})^m$. There are 165 conjugacy classes. In all 165 cases the group extends to a small resolution \mathcal{X} such that a weak Calabi–Yau model can be obtained by a resolution of \mathcal{X}/G . There are 144 classes that admit a projective Calabi–Yau model that can be obtained as described in the introduction. This means that the group G extends to a not necessarily minimal projective resolution $\tilde{\mathcal{X}}$ such the $\tilde{\mathcal{X}}/G$ admits a resolution in the form of a (projective) Calabi–Yau manifold. It is known that the Euler number and divisor class number are birational invariant for weak Calabi–Yau manifolds (since they are related by flops.) Therefore we always can use a small minimal resolution $\tilde{\mathcal{X}}$ (also a non-projective one) for the computation of the Euler and divisor class number. This actually means that we sometimes compute the Hodge numbers for a group with projective Calabi–Yau model using a non-projective model.

Computation of the divisor class number

So let $G \subset \bar{\mathcal{H}}$ be a subgroup that is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^m$. We choose a small resolution $\tilde{\mathcal{X}}$ such that G extends. We have to determine the divisor class number of a Calabi–Yau resolution of $\tilde{\mathcal{X}}/G$. This is the sum of the divisor class number of \mathcal{X}/G and the number of exceptional divisors. The divisor class number of \mathcal{X}/G can be computed since we know the representation of $\bar{\mathcal{H}}$ on $\mathrm{Cl}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$. This can be done by a program. To get the number of exceptional divisors we need information about the singularities of \mathcal{X}/G . Locally they are of the form \mathbb{C}^3/H with an abelian group $H \subset \mathrm{SL}(3, \mathbb{Z})$, where all elements of H have order ≤ 2 . After diagonalization H consists of sign changes. Since the determinants are one we have an even number of sign changes. In this cases the resolution is easy to produce (see [FS1]). The result is that the singular locus of \mathbb{C}^3 consists of one line or of two crossing lines. In the first case we have one exceptional divisor in the resolution in the second case we have two. This shows that the number of exceptional divisors of a (weak) Calabi–Yau resolution of $\tilde{\mathcal{X}}/G$ equals the number of components of the fixed point locus. We want to

express this in the singular model \mathcal{X} . So let $C \cong \mathbb{P}^1$ be an exceptional curve that projects to a node in \mathcal{X} . Then there is an $g \in G$, $g \neq e$, that fixes the node. This node must be an isolated fixed point of g since otherwise the fixed locus on $\tilde{\mathcal{X}}$ would be not smooth. So we see the following lemma.

8.1 Lemma. *Let $G \subset \bar{\mathcal{H}}$ be a subgroup of type $(\mathbb{Z}/2\mathbb{Z})^m$. Let a be the number of G -equivalence classes of irreducible curves in \mathcal{X} that are in the fixed point locus and let b be the number of G -equivalence classes of nodes that are isolated fixed points. Then the divisor class number of a weak Calabi–Yau model of \mathcal{X}/G equals*

$$a + b + \dim(\mathrm{Cl}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q})^G.$$

Computation of the Euler number

Our main tool will be the string theoretic formula [Ro]:

Let X be a weak Calabi–Yau three fold and G a finite group of biholomorphic transformations that leave the Calabi–Yau three form invariant. Assume that the stabilizers G_a are contained in the group $\mathrm{SL}()$ of the tangent space. Then the Euler number of a resolution of X/G in the form of a Calabi–Yau manifold equals

$$e(M, G) = \frac{1}{\#G} \sum_{gh=hg} e(M^{(g,h)}).$$

Here $M^{(g,h)}$ denotes the common fixed point set of g, h .

Let $G \subset \bar{\mathcal{H}}$ be a subgroup that extends to a small resolution $\tilde{\mathcal{X}}$. Then the string theoretic formula applies. Recall that the fixed point locus of G on $\tilde{\mathcal{X}}$ is a curve C with smooth irreducible components. Let S be the singular locus of S (crossing points). Then $e(C) = e(C - S) + \#S$. The Euler number of $C - S$ is the sum of the Euler numbers of its connected components. The Euler number of a smooth non-compact curve K with compactification \bar{K} is just

$$e(K) = 2 - 2g(\bar{K}) - \#(\bar{K} - K) \quad (g(\bar{K}) = \text{genus}).$$

Since we know the equations of \mathcal{X} it is no problem to compute the fixed point loci on \mathcal{X} . This is the image of C in \mathcal{X} . We know already that the fixed point loci of elements of order two consist of (smooth) rational and elliptic curves and isolated points (nodes). Hence we have just to analyze what happens on a exceptional \mathbb{P}^1 in $\tilde{\mathcal{X}}$. This is clarified by the following two lemmas.

8.2 Lemma. *Let $g \in \bar{\mathcal{H}}$ an element of order two which fixes a node a . There are two cases:*

- 1) *a is an isolated fixed point of g . In this case g extends as identity on the exceptional \mathbb{P}^1 over a .*
- 2) *a is not an isolated fixed point of g . Then g has precisely two fixed points on the exceptional \mathbb{P}^1 . These two fixed points are intersection points of two further fixed point curves of g which are visible already in \mathcal{X} .*

This lemma follows from our investigations of the involutions during the proof of 7.3. \square

8.3 Lemma. *Let g, h be two different commuting elements of $\bar{\mathcal{H}}$ of order two. Assume that they fix a joint node a . Then there are two possibilities for the joint fixed point locus on the exceptional \mathbb{P}^1 over a .*

- 1) *It consists of two points. This happens if a is an isolated fixed point of one of the three $\{g, h, gh\}$.*
- 2) *It is empty. This happens if it is not an isolated fixpoint of any of the three $\{g, h, gh\}$.*

Proof. We first mention that it cannot happen that a is an isolated fixed point of two of the g, h, gh . The reason is that there is only one conjugacy class of elements of order two which fix a node as isolated fixed point (the conjugacy class of σ_2). This conjugacy class consists of 6 elements and each of them fixes 16 nodes and nothing else. The 6 blocks of 16 nodes are pairwise disjoint and exhaust all $96 = 6 \cdot 16$ nodes.

Case 1) is clear, assume for example that gh has the node as an isolated fixed point. Then it acts as identity on the exceptional fibre. Then g and h are inverse on this fibre and the same fixed points. From 8.2 we know that they are two.

We treat the second case. From the assumption follows that g and h are different. We know that g has precisely two fixed points on the exceptional fibre over a . We choose the biholomorphic map $\mathbb{P}^1 \cong \bar{\mathbb{C}}$ such that g acts as $g(z) = -z$. Since h commutes with g it acts on the two fixed points. It cannot fix both since then h would be equal to g . Hence h permutes 0 and ∞ and the only possibilities are the transformations $h(z) = \pm 1/z$. But they have different fixed points. Hence the joint fixed point set on the \mathbb{P}^1 is empty. \square

We have collected all what we need for the computation of the Euler number. A computer calculation gives now the following list.

There are 40 different pairs (cl, e) of divisor class numbers and Euler numbers of weak Calabi–Yau manifolds that are produced by subgroups of $\bar{\mathcal{H}}$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^m$.

(28, 56), (20, 40), (16, 32), (14, 28), (15, 26), (12, 16), (10, 20), (8, 16), (4, 8),
 (26, 50), (10, 8), (22, 44), (14, 26), (18, 28), (13, 20), (44, 88), (11, 16), (16, 28),
 (34, 68), (20, 32), (16, 16), (14, 20), (13, 26), (22, 40), (15, 28), (41, 82), (26, 52),
 (6, 8), (12, 8), (17, 32), (19, 38), (70, 140), (14, 16), (12, 20), (10, 16), (9, 14),
 (46, 92), (40, 80), (32, 64), (18, 32).

There are 33 different pairs (h^{11}, h^{12}) of Hodge numbers which are produced by subgroups of $\bar{\mathcal{H}}$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^m$ and with a projective

model as described in the introduction.

$$(14, 0), (12, 4), (20, 0), (14, 1), (6, 2), (26, 0), (32, 0), (26, 1), (70, 0), (18, 2), \\ (16, 8), (44, 0), (15, 2), (15, 1), (12, 2), (10, 2), (14, 6), (9, 2), (4, 0), (22, 2), \\ (10, 0), (16, 0), (14, 4), (12, 8), (22, 0), (20, 4), (13, 0), (28, 0), (19, 0), (16, 2), \\ (34, 0), (40, 0), (8, 0).$$

The list seems to contain several examples that are not contained in the (physicists) literature (see [CD]).

9. Two more examples

We consider the group H of order 16 that we described in Theorem 7.2. There are transformation of order 3 in \bar{H} that normalize this group, for example

$$h_3 : (Y_0, -Y_3, iY_1, iY_2, X_3, X_1, X_0, X_2).$$

We consider now the group H_{48} of that is generated by this element and the group in Theorem 7.2. The order of this group is 48. Using Proposition 6.2 on can check that H_{48} extends to a projective small resolution $\tilde{\mathcal{X}} \rightarrow \mathcal{X}$. All 32 elements which are not contained in H are of order 3. The 16 subgroups of order 3 of H_{48} are conjugated under H . This means that the fixed point locus of h_3 in \mathcal{X} maps under $\mathcal{X} \rightarrow \mathcal{X}/H$ biholomorphic to the whole fixed point locus of H_{48}/H . Recall the the fixed point locus of h_3 does not contain a node and consists of an elliptic curve and 4 points. The H_{48} -invariant part of $\text{Cl}(\mathcal{X})$ has rank two. Hence the Picard number of a Calabi–Yau resolution of \mathcal{X}/H_{48} equals 8. The string theoretic formula applied to $\tilde{\mathcal{X}}/H, H_{48}/H$ gives $e = (1/3)(4 + 8 \cdot 4) = 12$. Hence we obtain a Calabi–Yau manifold with Hodge numbers

$$H_{48} : \quad h^{11} = 8, \quad h^{12} = 2 \quad (e = 12).$$

Our last example starts with the same group H but we extend it now by a involution

$$h_2 : (Y_2, iY_3, -iY_0, -Y_1, -iX_3, iX_2, -X_1, X_0)$$

This involution normalizes H . Hence we get a group H_{32} of order 32. Using Proposition 6.2 on can check that H_{32} extends to a projective small resolution $\tilde{\mathcal{X}} \rightarrow \mathcal{X}$. The whole fixed point locus of H_{32} consists of 24 elliptic curves and 32 rational curves. The image in \mathcal{X}/H (i.e. the fixed point locus of h_2 acting on \mathcal{X}/H consists of 3 elliptic curves and 2 rational curves. One obtains a (projective) Calabi–Yau manifold with Hodge numbers

$$H_{32} : \quad h^{11} = 7, \quad h^{12} = 3 \quad (e = 8).$$

Final Remark. With a refinement of the described methods we could compute 1 344 of the 4 117 cases the Euler number where 786 admit a projective Calabi–Yau model. The Euler numbers of weak Calabi–Yau models that we got are

2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 38, 40, 44, 46, 50, 52, 56, 58, 64, 68, 70, 80, 82, 88, 92, 100, 140.

Euler numbers of a projective Calabi–Yau model from this list are

4, 8, 12, 14, 16, 20, 24, 26, 28, 32, 38, 40, 44, 50, 52, 56, 64, 68, 80, 88, 100, 140.

The Euler numbers, we could get, are all positive. This is clear from the formula for string theoretic Euler number in case that the group extends to a small resolution. So we cannot get mirror pairs.

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